Name: Answer Key

MA 131 Test 2 Form A

1. In this problem, consider some function $f(x)$. Be as precise as possible when answering the following questions.

(a) If you were to explain to someone in plain English what is meant by the derivative of $f(x)$, what would you tell them?

The derivative of $f(x)$ is the rate of change of $f(x)$

(b) Give an explanation of what $f'(x)$ is using tangent lines.

$f'(x)$ is the slope of the tangent line to $f(x)$ at the point $(x, f(x))$

(c) Give the official, mathematical definition of the derivative of $f(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. From 1940 to 1990, the number of parking tickets given out in Washington DC increased every year, but each year's increase was smaller than the previous year's increase.

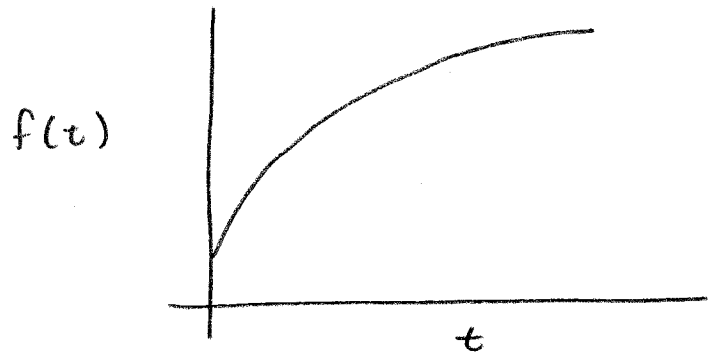
(a) If $f(t)$ is the number of parking tickets given out in Washington D.C. in year t, where t is between 1940 and 1990. Is $f'(t)$ positive or negative? Is $f''(t)$ positive or negative?

$f'(t) > 0$ since # tickets is increasing

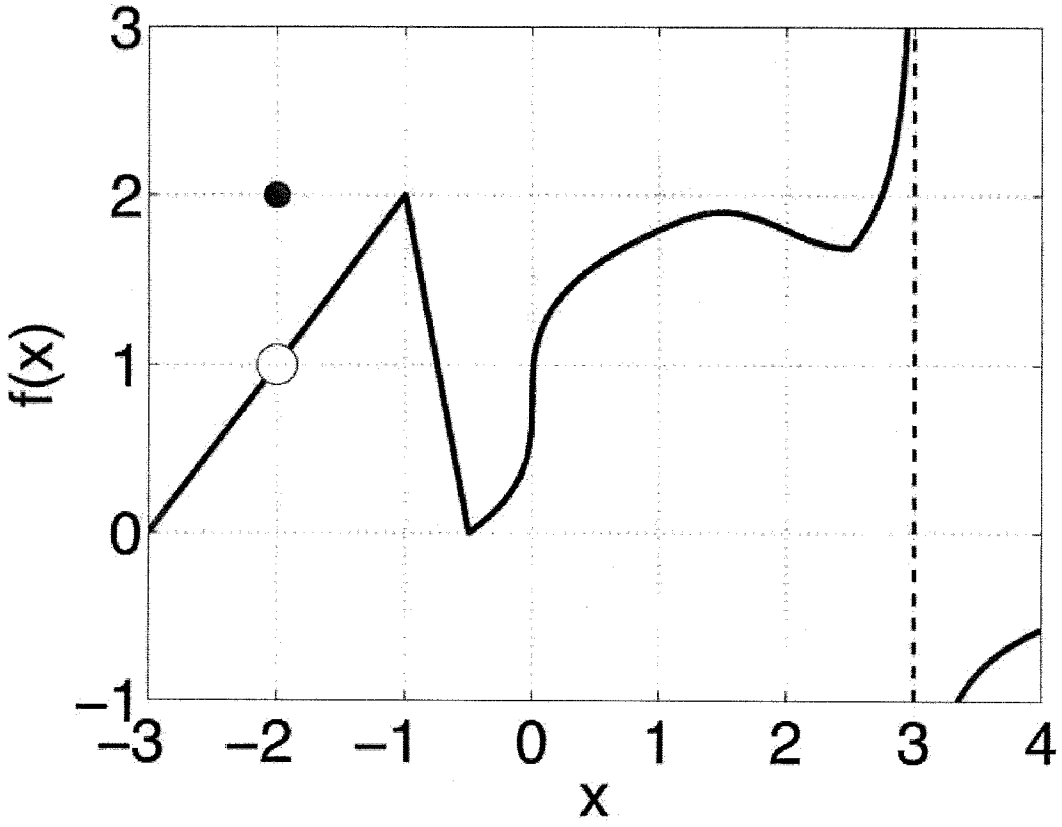
$f''(t) < 0$ since rate of increase is decreasing

(b) Draw a rough graph of $f(t)$. Your graph will be graded on two things: (1) The 1st derivative of the curve you draw must have the correct sign and (2) The 2nd derivative of the curve you draw must have the correct sign.

Graph should be increasing and concave down



3. Consider the graph below



For each of the following four points (part (a) through part (d)), tell me (1) is $f(x)$ differentiable there? (2) Is $f(x)$ continuous there? If your answer is no to either of these questions, explain why.

(a) $x = -2$

Not continuous since $\lim_{x \to -2} f(x) \neq f(-2)$

Can't be differentiable since its not continuous

(b) $x = -1$

Continuous

Not differentiable due to sharp point

(c) $x = 0$

Continuous

Not differentiable due to vertical tangent line

(d) $x = 2$

Continuous & differentiable

4. In the space below, draw a graph of the following function:

$$f(x) = \frac{x^3}{3} - 36x + 472\pi^7$$

To help you draw your graph, answer the following questions:

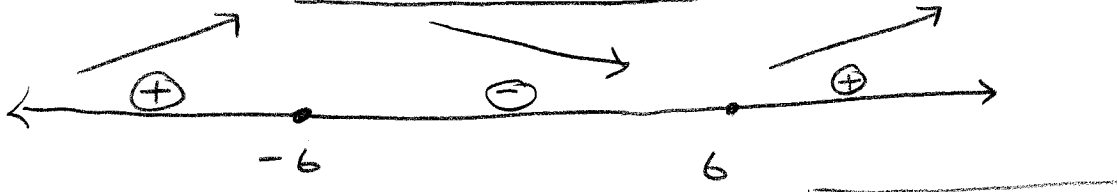
1. Where are the relative maxima and relative minima?
2. Where is $f(x)$ increasing and decreasing?
3. Where is $f(x)$ concave up/concave down?

Do NOT worry about what the exact y-values are for your graph. On your graph, clearly mark the x-values of your critical points and draw a curve that is increasing/decreasing and concave up/down in the right places.

$f'(x) = x^2 - 36$

$0 = x^2 - 36 = (x-6)(x+6)$

$x = 6 \qquad x = -6$ ← Critical Points



$f'(-7) = (-13)(-1) = 13 > 0$

$f'(0) = (-6)(6) = -36 < 0$

$f'(7) = (1)(13) = 13 > 0$

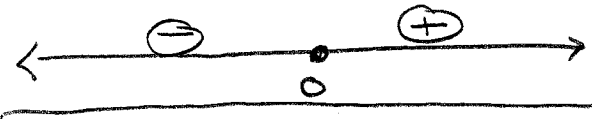
Rel Max at $x = -6$

Rel Min at $x = 6$

Increasing: $(-\infty, -6) \cup (6, \infty)$

Decreasing: $(-6, 6)$

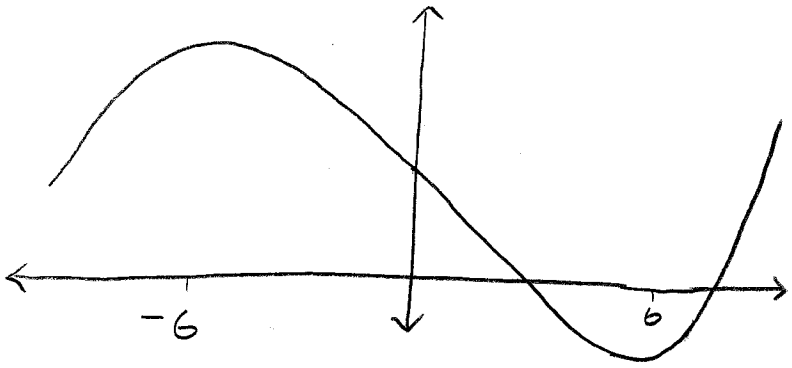
$f''(x) = 2x$ $\qquad 0 = 2x$ $\qquad 0 = x$



Inflection Pt at $x = 0$

Concave Down: $x < 0$

Concave Up: $x > 0$



5. You are operating a hotel, and from past data you have collected you know that the number of rooms you rent, n, is related to the price you charge per room p, by the following relationship:

$$n = 200 - p$$

What price should you charge per room to make your revenue as large as possible? Assume that p is between 0 and 200. Hint: revenue = number of rooms rented × price charged.

R = revenue

n = # of rooms rented

p = price charged per room

$R = np$ ← objective Eq

$n = 200 - p$ ← constraint

$R = (200 - p)p = 200p - p^2$

$R' = 200 - 2p$

$0 = 200 - 2p$

$2p = 200$

$p = 100$

$p = 100$ is critical pt of R. To make sure its a max, find

$R(p=0)$, $R(p=100)$, $R(p=200)$

bounds given in problem

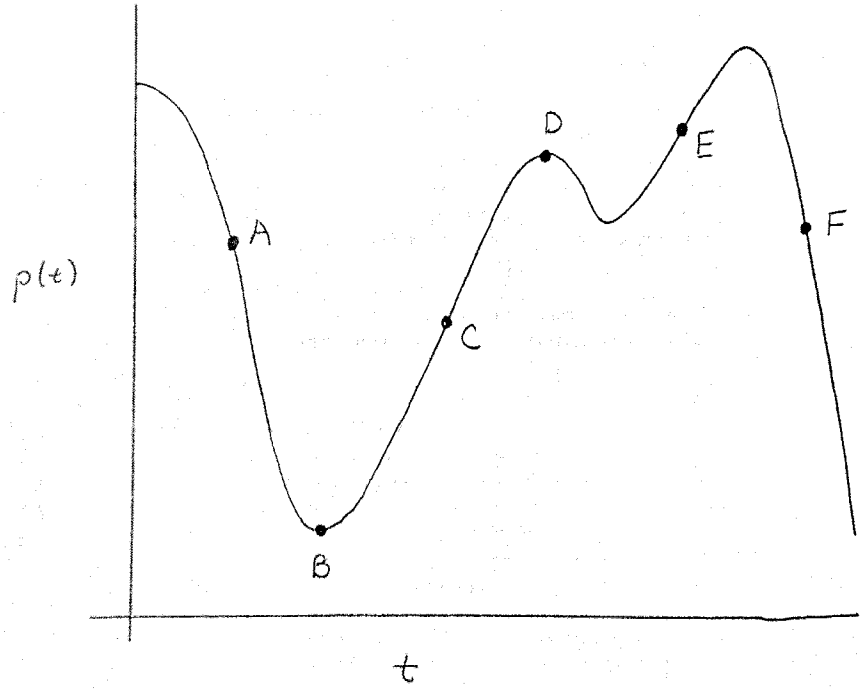
$R(p=0) = 0$

$R(p=100) = 100 \cdot 100 = 100^2$

$R(p=200) = 0$

Revenue maximized by charging $100/night

6. A car is moving along a straight road. The position of the car, $p(t)$, is graphed below (t = time). Answer questions (a) through (e) about the movement of the car: (Hint: velocity is the derivative of position, acceleration is the 2nd derivative of position. Velocity means how fast the car is moving.)



To receive full credit, give explanations for your answers.

(a) Is the car moving faster at point D or point E?

E. Slope of tangent line greater at E.

(b) How fast is the car moving at point B?

Not moving. Tangent line horizontal so $v = 0$

(c) Is the acceleration positive or negative at point B?

Positive. $a = p'' > 0$ at B since v concave up at B

(d) Which direction is the car moving at point F?

Backwards. Slope of tangent line is negative

(e) Is the velocity increasing or decreasing at point D?

Decreasing since $v' = p'' < 0$ at D since graph of p concave down at D

7. You are building a rectangular fence. One side of the fence will be facing the road, and city laws require you to use a fencing material that costs \$15 per linear foot. For the other three sides, you can use a cheaper fencing material that costs only \$5 per linear foot. If you only have \$400 total to build the fence, how should you construct the fence in a way to make the area inside the fence as large as possible?

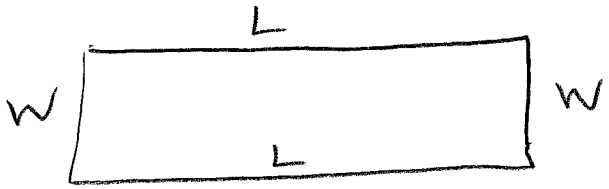
If you are having trouble with this problem, you may find it helpful to break things down into smaller steps:

1. Draw a picture and assign variable names to all quantities.
2. Write the objective equation.
3. Write the constraint equation.
4. Use the constraint equation to turn the objective equation into a function of one variable.
5. Find the maximum of the objective equation. Make sure you justify WHY it is the point you have found is a maximum of the objective equation.

① A = area inside fence

L = length of rectangle

W = width of rectangle



② Objective Eq: $A = LW$

③ Constraint: $400 = 15L + 5W + 5L + 5W$

$400 = 20L + 10W$

④ $10W = 400 - 20L$

$W = 40 - 2L$

$A = L(40 - 2L)$

$A = 40L - 2L^2$

⑤ $0 = A' = 40 - 4L$

$4L = 40$

$L = 10$

$W = 40 - 2L = 40 - 20 = 20$

Area maximized by setting $L = 10$, $W = 20$

To make sure this is a max, find $A(L=0)$, $A(L=10)$, $A(L=20)$

$A(L=0)$: smallest possible length

$A(L=20)$: largest possible length

$A(L=0) = 0$

$A(L=20) = 0$

$A(L=10) = 10 \cdot 20$

$= 200$ ✓

max at $L=10$

Honor Pledge: I have neither given nor received unauthorized aid on this test.

Signature: _____

Bonus (5 pts): Find the derivative of the following. Be sure to show all of your work.

$$y = 13\frac{(x+2)(x+7)}{2x+(x+\sqrt{x^2+1})^{3/2}}$$

$y = 13\dfrac{f(x)}{g(x)}$ $\qquad f(x) = (x+2)(x+7) = x^2+9x+14$

$g(x) = 2x + (x+\sqrt{x^2+1})^{3/2}$

$f'(x) = 2x + 9$

$g'(x) = 2 + \frac{3}{2}(x+\sqrt{x^2+1})^{1/2}(1 + \frac{1}{2}(x^2+1)^{-1/2}2x)$

$y' = 13\dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$13\Big\{(2x+(x+\sqrt{x^2+1})^{3/2})(2x+9) - (x^2+9x+14)(2+\tfrac{3}{2}(x+\sqrt{x^2+1})^{1/2}(1+x(x^2+1)^{-1/2}))\Big\}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\left(2x+(x+\sqrt{x^2+1})^{3/2}\right)^2$$