Name: Answer Key

MA 131 Test 3 Form B

1. (24 Points) Find the derivative of the following functions. Be sure to show your work.

(a) $f(t) = \dfrac{3t^2}{t^3 + 15t}$ Using quotient rule

$$f'(t) = \frac{(t^3+15t)(6t) - 3t^2(3t^2+15)}{(t^3+15t)^2}$$

$$= \frac{-3t^4+45t^2}{(t^3+15t)^2}$$

(b) $y = \ln(2x^2 + x^4 + 7)$ $y = \ln g(x)$ $y' = \dfrac{g'(x)}{g(x)}$

$$y' = \frac{4x + 4x^3}{2x^2 + x^4 + 7}$$

(c) $g(x) = x^2\sqrt{x^2+1}$

Product Rule

$$g'(x) = 2x\sqrt{x^2+1} + x^2\left(\tfrac{1}{2}\right)(x^2+1)^{-1/2}(2x)$$

$$= 2x\sqrt{x^2+1} + \frac{x^3}{\sqrt{x^2+1}}$$

(d) $y = e^{x^2\sqrt{x^2+1}}$

$$y' = \left(2x\sqrt{x^2+1} + \frac{x^3}{\sqrt{x^2+1}}\right)e^{x^2\sqrt{x^2+1}}$$

2. **(16 Points)** Salt is poured out of a funnel onto a table, and the salt piles up in a conical pile whose height is roughly 0.6 times its radius. This means that

$$V(t) = \frac{1}{5}\pi\,[r(t)]^3$$

where $V(t)$ is the volume and $r(t)$ is the radius. If the volume is increasing at a rate of 4 cm^3/s, how fast is the radius of the cone increasing when $r = 5$?

$$V'(t) = \frac{3}{5}\pi\,[r(t)]^2 \frac{dr}{dt}$$

$$V'(t) = 4, \quad r(t) = 5$$

$$4 = \frac{3}{5}\pi\, 5^2 \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{4}{15\pi}}$$

3. **(16 Points)** 10,000 bacteria are placed in a culture. Let $B(t)$ be the number of bacteria present in the culture after t hours have passed, and suppose that $B(t)$ satisfies the differential equation

$$B'(t) = \frac{1}{4}B(t)$$

(a) Write a formula for $B(t)$.

$$B(t) = 10{,}000\, e^{t/4}$$

(b) How many bacteria are there after 4 hours?

$$B(4) = 10{,}000\, e^{4/4} = \boxed{10{,}000e}$$

(c) How fast is the bacteria culture growing when there are 50,000 bacteria in the culture?

$$B' = \frac{1}{4}\, 50{,}000 = \boxed{12{,}500 \frac{\text{bacteria}}{\text{hr}}}$$

(d) How many bacteria are in the culture when it is growing at a rate of 20,000 bacteria per hour?

$$20{,}000 = \frac{1}{4} B$$

$$\boxed{80{,}000 \text{ bacteria}}$$

4 (12 Points). In this problem, consider some function $f(x)$. Be as precise as possible when answering the following questions.

(a) If you were to explain to someone in plain English what is meant by the derivative of $f(x)$, what would you tell them?

The derivative is the rate of change of $f(x)$

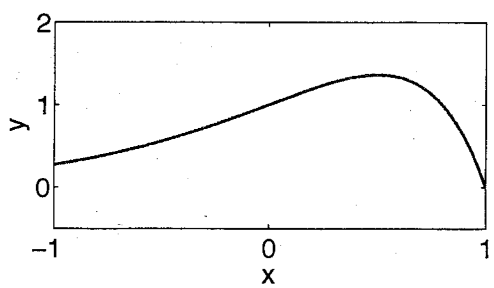
(b) Give an explanation of what $f'(x)$ is using tangent lines.

$f'(x)$ is the slope of the tangent line at the point $(x, f(x))$

(c) Give the official, mathematical definition of the derivative of $f(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5. (16 Points) The graph of $y = (1-x)e^{2x}$ is displayed below and contains one local maximum. Find the x-value of this local maximum.



$$y' = (-1)e^{2x} + (1-x)(2)e^{2x}$$

$$= (-1 + 2 - 2x)e^{2x}$$

$$= (1-2x)e^{2x}$$

$$0 = (1-2x)e^{2x}$$

$$0 = 1 - 2x$$

$$2x = 1$$

$$\boxed{x = 1/2}$$

Normally, you should use 1st/2nd derivative to figure out if this critical point is actually a max, Here you know its a max from the graph

6. (16 Points) The president of a country decides to start a war to divert attention from a scandalous event in his or her personal life. Unfortunately, his plan is foiled and the media reports his plan to the world. The number of people (in millions) who have heard this news report, $P(t)$, obeys the following function:

$$P(t) = 7000\left(1 - e^{-20t}\right)$$

(a) How many people have heard the news at time $t = 0$?

$P(0) = 7000(1 - e^{-20\cdot 0}) = 7000(1-1) = 0$

(b) As time goes by, is the number of people who have heard the news increasing or decreasing? Why?

$P(t) = 7000 - 7000e^{-20t}$

$P'(t) = 140{,}000e^{-20t} > 0$

Increasing since $P'(t) > 0$

(c) As time goes by, is the *rate of increase* of $P(t)$ getting bigger or smaller?

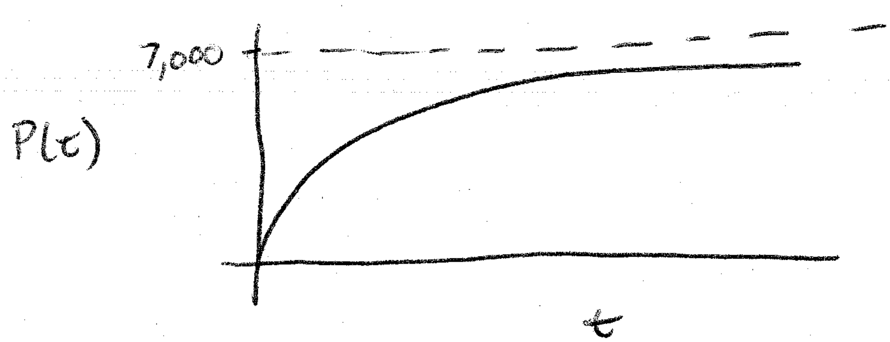
$P''(t) = 140{,}000(-20)e^{-20t} < 0$

$P'(t)$ Decreasing since $P''(t) < 0$

Rate of increase getting smaller

(d) Using the information from parts (b) and (c), draw a sketch of the graph of $P(t)$. Before you draw your graph, think about what happens to the derivative of $P(t)$ as t grows larger and larger, and incorporate this fact into your graph.

As t gets big, $e^{-20t} = \frac{1}{e^{20t}}$ gets close to 0 so we expect $P(t)$ to become roughly constant



Bonus Question: (5 Points) In class, we mentioned the idea of exponential growth as a potential model for a population of rabbits. Specifically, we said that the population of rabbits, $P(t)$, is a solution to the following differential equation:

$$P'(t) = 2P(t)$$
$$P(0) = 10$$

As we showed in class, the solution to this equation is $P(t) = 10e^{2t}$. The problem with this model is that it predicts that the rabbit population will never stop growing; $P(1) \approx 74, P(2) \approx 546, P(10) \approx$ 4.8 billion, and $P(20) \approx 2.3$ quintillion. This is obviously wrong. In real life, the rabbit population will never get close to 2.3 quintillion. The reason for this is that there is only a finite amount of resources (food, etc.), so eventually, the rabbit population growth will slow and eventually stop. A better model that accounts for this limited amount of resources is the *logistic growth* model:

$$P'(t) = \frac{r}{K}P(t)\left[K - P(t)\right]$$
$$P(0) = P_0$$

r and K are positive constants, r is called the *growth rate*, K is called the *carrying capacity*, and $P(t)$ is the size of the population at time t.

(a) For what values of $P(t)$ is the population size increasing? For what values of $P(t)$ is the population size decreasing?

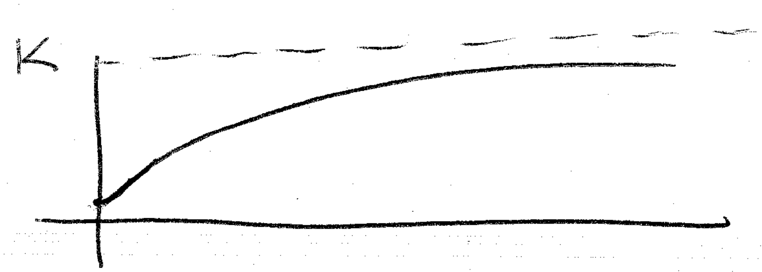
Assuming P not negative

Increasing for $P(t) < K$

Decreasing for $P(t) > K$

(b) Suppose $P_0 = 2, r = 1$, and $K = 1000$. Use the differential equation above to sketch a possible graph of $P(t)$. Be sure to mark K on your y-axis.

Expect graph to be increasing



(Continued on next page)

(c) Suppose $P_0 = 2000, r = 1$, and $K = 1000$. Use the differential equation above to sketch a possible graph of $P(t)$. Be sure to mark K on your y-axis.

Expect $P(t)$ to be decreasing
since $P_0 > K$