Name: Answer Key

MA 131 Test 4 Form A

1. a) Which of the following is the *antiderivative* of $f(x) = \ln(x)$? Why? **[10 points]**

(a) $\frac{1}{x} + C$ (b) $x \cdot \ln(x) - x + C$ (c) $\frac{1}{2} \cdot (\ln(x))^2 + C$

$\frac{d}{dx}\left(\frac{1}{x} + c\right) = \frac{d}{dx} x^{-1} = -x^{-2} = \frac{-1}{x^2} \neq \ln x$ NOT (a)

$\frac{d}{dx}(x\ln x - x + c) = x\left(\frac{1}{x}\right) + \ln x - 1 = 1 + \ln x - 1 = \ln x$ ✓

$\frac{d}{dx}\left[\frac{1}{2}(\ln x)^2 + c\right] = \frac{2}{2}\ln x \frac{d}{dx}\ln x = \frac{\ln x}{x}$ NOT (c)

$\boxed{x\ln x - x + c}$ is antiderivative of $\ln x$

since $\frac{d}{dx}(x\ln x - x + c) = \ln x$

b.) Find the following definite integral. Evaluate all natural logs and write your final answer as a single whole number. **[8 points]**

$$\int_1^e \ln(x)\, dx$$

Use Fundamental Theorem of Calculus

$\int_1^e \ln x\, dx = x\ln x - x \Big|_1^e$

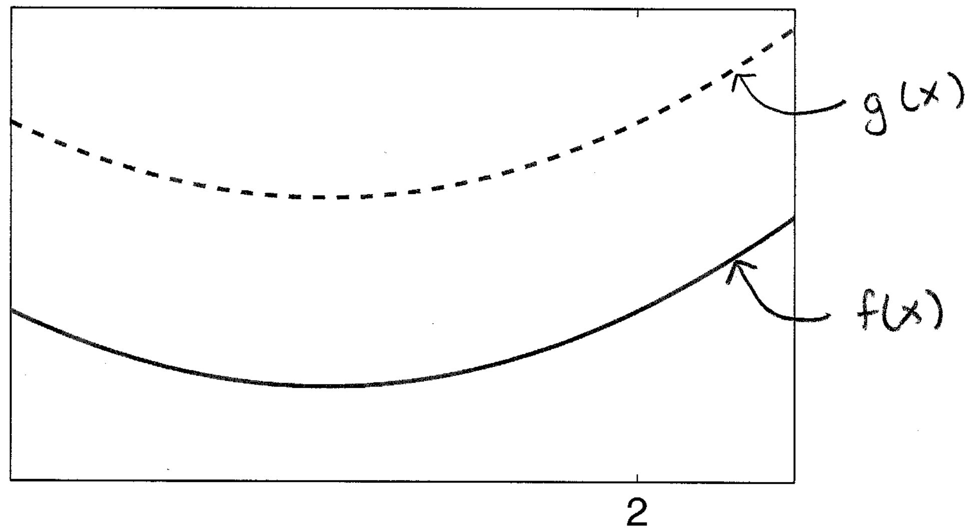
$= e\ln e - e - (1\ln 1 - 1)$

$= \cancel{e(1)} \cancel{-e} - 1\ln 1 + 1$

$= -1(0) + 1$

$= \boxed{1}$

2.) In the graph below, the function $g(x)$ was obtained by shifting the graph of $f(x)$ up by 1. In the graph below, $g(x)$ is the dashed line, and $f(x)$ is the solid line



If $f'(2) = 2$, what is $g'(2)$? Why? **[8 points]**

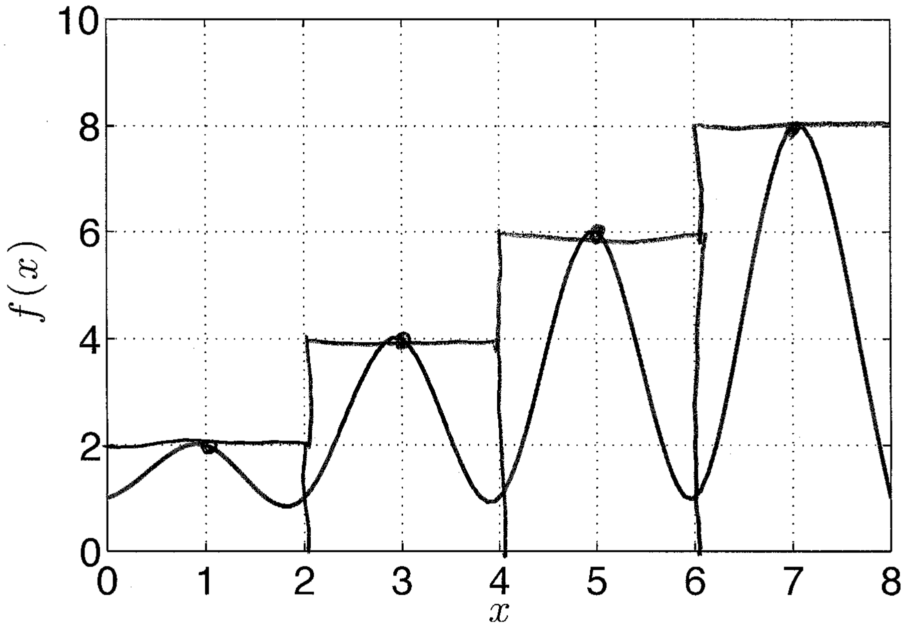
$g(x) = f(x) + 1$

$g'(x) = f'(x)$

$\boxed{g'(2) = f'(2) = 2}$

Shifting graph up or down doesnt change its derivative

3.) Below is the graph of some function $f(x)$.



a.) Approximate $\int_0^8 f(x)\,dx$ with a Riemann sum. Break the interval $[0, 8]$ into 4 subintervals, and use the function value of $f(x)$ at the *midpoint* of each interval to determine the height of each rectangle. **[10 points]**

$$\int_0^8 f(x)dx \approx 2 \cdot 2 + 2 \cdot 4 + 2 \cdot 6 + 2 \cdot 8$$

$$= 4 + 8 + 12 + 16$$

$$= 12 + 28$$

$$= \boxed{40}$$

b.) Is the approximation you obtained in part (a): larger than $\int_0^8 f(x)\,dx$, smaller than $\int_0^8 f(x)\,dx$, or exactly equal to $\int_0^8 f(x)\,dx$. Why? **[4 points]**

Approximation is bigger than $\int_0^8 f(x)dx$ since there is more area in the rectangles than there is under the curve $f(x)$

4.) A rocket is launched from the ground, and its velocity as a function of time $v(t) = -t^2 + 16$ (t is in seconds, $v(t)$ is in ft/sec). If $v(t)$ is positive, then this indicates that the rocket is moving upwards, and a negative velocity indicates that the rocket is moving down towards the ground.

a.) When is the rocket moving upwards? Downwards? **[6 points]**

$0 > -t^2 + 16$

$t^2 > 16$

$t > 4$

Downwards when $t > 4$

$0 < -t^2 + 16$

$t^2 < 16$

$t < 4$

Upwords when $t < 4$

b.) At what value of t does the rocket reach its peak height? **[4 points]**

$\boxed{t = 4}$

c.) Compute the area under the velocity curve from time $t = 0$ to time $t = t^*$, where t^* is the answer you found for part (b). For example, if your answer to part (b) was 2, find the area under the velocity curve from time $t = 0$ to time $t = 2$. **[10 points]**

$$\int_0^4 (-t^2 + 16)\,dt = -\frac{t^3}{3} + 16t \Big|_0^4$$

$$= -\frac{4^3}{3} + 16(4) - 0$$

$$= \frac{-64}{3} + 64$$

$$= \frac{-64 + 192}{3}$$

$$= \boxed{\frac{128}{3}}$$

d.) What does your answer to part (c) represent about the rocket? DO NOT just say it's the area under the velocity curve. **[6 points]**

The maximum height of the rocket is $\frac{128}{3}$ ft

5.) Find the volume of the solid obtained by revolving $f(x) = x\sqrt{x^3+1}$ around the x-axis from $x = 0$ to $x = 1$. **[16 points] Bonus: [1 point]** Sketch this solid.

$$V = \pi \int_0^1 \left(x\sqrt{x^3+1}\right)^2 dx$$

$$= \pi \int_0^1 x^2(x^3+1)\,dx$$

$u = x^3 + 1$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

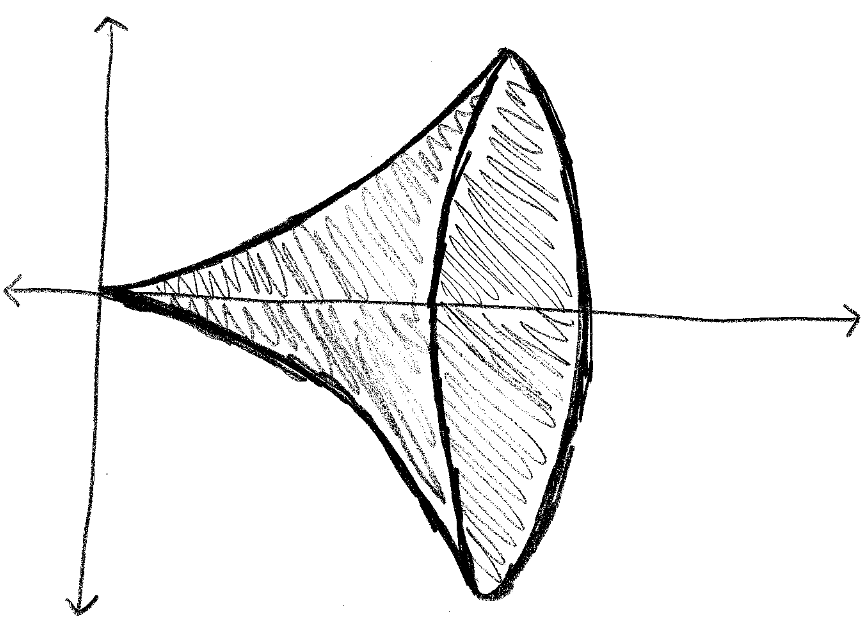
$x = 0, \quad u = 0^3 + 1 = 1$

$x = 1, \quad u = 1^3 + 1 = 2$

$$= \frac{\pi}{3} \int_1^2 u\,du$$

$$= \frac{\pi}{3} \frac{u^2}{2} \Big|_1^2$$

$$= \frac{\pi}{3}\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \frac{\pi}{3}\left(2 - \frac{1}{2}\right) = \frac{\pi}{3}\left(\frac{3}{2}\right) = \boxed{\frac{\pi}{2}}$$



x	$f(x)$
0	0
1	$\sqrt{2} \approx 1.4$
2	$2\sqrt{9} = 6$

6.) Water is flowing into a tank at a rate of $r(t)$ gallons per hour, where $r(t) = 10 - t$.

a.) How much water flows into the tank from time $t = 0$ to time $t = 4$? **[10 points]**

$$\int_0^4 (10 - t)\,dt = 10t - \frac{t^2}{2}\Big|_0^4$$

$$= 10(4) - \frac{4^2}{2} - \left(10(0) - \frac{0^2}{2}\right)$$

$$= 40 - \frac{16}{2}$$

$$= 40 - 8$$

$$= \boxed{32 \text{ gallons}}$$

b.) What is the *average* rate at which the water flows into the tank from $t = 0$ to $t = 4$? **[8 points]**

$$\underline{\text{Avg Rate}} = \frac{1}{4-0}\int_0^4 (10-t)\,dt$$

$$= \frac{1}{4}\,32$$

$$= \boxed{8\ \frac{\text{gallons}}{\text{hr}}}$$

Bonus #1: Consider the following function $g(x)$:

$$g(x) = \int_0^x f(t)\,dt$$

If $f(10) = 5$, what is $g'(10)$? **[0 points]** Explain why. **[3 points]**

Let $F'(t) = f(t)$

$$g(x) = \int_0^x f(t)\,dt = F(x) - F(0)$$

$$g'(x) = F'(x) = f(x)$$

$$\boxed{g'(10) = f(10) = 5}$$

Bonus #2: An integral with ∞ or $-\infty$ in one of the bounds is called an *improper integral.* We make sense of these improper integrals by defining them as a limit of definite integrals. For example, the integral

$$\int_1^\infty f(x)\,dx$$

is defined as

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$$

Sometimes, definite integrals do not exist (this is similar to how certain limits do not exist). Consider the following two improper integrals. Determine the value of each integral (if it exists).

a.) [1 point] $\int_1^\infty \frac{1}{x}\,dx$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1)$$

$$= \lim_{b\to\infty} \ln b \leftarrow \boxed{\text{DNE}}$$

b.) [1 point] $\int_1^\infty \frac{1}{x^2}\,dx$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left(-\frac{1}{x}\Big|_1^b\right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right)$$

$$= 1 - \lim_{b\to\infty} \frac{1}{b}$$

$$= \boxed{1}$$